



TITLE:

The Gradient Theory of the Phase Transitions in Cahn-Hilliard Fluids with the Dirichlet boundary conditions(Mathematical Analysis of Phenomena in Fluid and Plasma Dynamics)

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CITATION:

ISHIGE, KAZUHIRO. The Gradient Theory of the Phase Transitions in Cahn-Hilliard Fluids with the Dirichlet boundary conditions(Mathematical Analysis of Phenomena in Fluid and Plasma Dynamics). 数理解析研究所講究録 1994, 862: 219-230

ISSUE DATE:

1994-03

URL:

<http://hdl.handle.net/2433/83851>

RIGHT:

The Gradient Theory of the Phase Transitions in Cahn-Hilliard Fluids with the Dirichlet boundary conditions

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1. Introduction

In this note we will investigate the asymptotic behavior of minimizer $\{u_\epsilon\}_{\epsilon>0}$ (as $\epsilon \rightarrow 0$) of the following variational problem :

$$(P_\epsilon) \quad \inf \left\{ \int_{\Omega} [\epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u)] dx \mid u \in W^{1,2}(\Omega; \mathbb{R}^n), u = g \text{ on } \partial\Omega \right\},$$

where Ω is a bounded domain in \mathbb{R}^N with C^2 smooth boundary $\partial\Omega$ and g is a Lipschitz continuous function from $\partial\Omega$ into \mathbb{R}^n . Here $W(x, \cdot)$ is a nonnegative continuous function which has two potential wells with equal depth. This type of problem is related to the study of the phase transitions of the Cahn-Hilliard fluids. See [8] and [9].

In [7] R.V.Kohn & P.Sternberg conjectured that minimizer of the variational problem, which is special case of (P_ϵ) ,

$$(SP_\epsilon) \quad \inf \left\{ \int_{\Omega} [\epsilon |\nabla u|^2 + \frac{1}{\epsilon} (u^2 - 1)^2] dx \mid u \in W^{1,2}(\Omega), u|_{\partial\Omega} = g \right\}$$

converges to a solution of

$$\inf \left\{ \frac{8}{3} P_{\Omega} \{u = 1\} + 2 \int_{\partial\Omega} |d(u) - d(g)| d\mathcal{H}_{N-1} \mid u \in BV(\Omega), |u| = 1 \text{ a.e.} \right\},$$

where $d(t) = \int_{-1}^t |s^2 - 1| ds$. Here \mathcal{H}_{N-1} is the $N - 1$ dimensional Hausdorff measure.

In this note, we will study the asymptotic behavior of minimizer of (P_ϵ) , and as a byproduct, we will state the affirmative results to the conjecture in [7].

Recently using the theory of Gamma-convergence, several authors studied the asymptotic behavior of the minimizer of the problem:

$$(E_\epsilon) \quad \inf \left\{ \int_{\Omega} [\epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(u)] dx \mid u \in W^{1,2}(\Omega; \mathbb{R}^n), \int_{\Omega} u(x) dx = m \right\},$$

where m is a constant vector in \mathbb{R}^n . For the scalar case (i.e. $n = 1$), see [8] and [9]. For the vector case (i.e. $n \geq 2$), see [1] and [4]. Our results on the problem (P_ϵ) depend mainly on the study of asymptotic behavior of minimizer of (E_ϵ) . However there are several different aspects between the asymptotic behavior of minimizer of (P_ϵ) and that of (E_ϵ) . In fact, minimizer of (E_ϵ) generates the only interior layer, but minimizer of (P_ϵ) generates both the interior and the boundary layers as $\epsilon \rightarrow 0$.

On the other hand, we can easily see that minimizer of (SP_ϵ) satisfies the equation:

$$(CP_\epsilon) \quad \begin{cases} \epsilon^2 \Delta u - u(u-1)(u+1) = 0 & \text{in } \Omega, \\ u(x) = g(x) & \text{on } \partial\Omega. \end{cases}$$

Then there exist several results for the solutions of (CP_ϵ) obtained by using the method of matched expansion. Our results also seem to be closely related to [2] and [3].

We will give the precise conditions of the functions $W(x, u)$ and $g(x)$. Let $W(x, u) : \overline{\Omega} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous nonnegative function, and for any $x \in \overline{\Omega}$ $W(x, u) = 0$ if and only if $u = \alpha$ or β . Here we note α and β are constant vectors independent of x . We assume that there exist two constants K_1 and K_2 such that

$$(1.1) \quad \sup_{u \in \partial[K_1, K_2]^n} W(x, u) \leq W(x, v) \quad \text{for all } x \in \overline{\Omega}, v \notin [K_1, K_2]^n$$

and

$$(1.2) \quad g(x) \in [K_1, K_2]^n \quad \text{for all } x \in \partial\Omega.$$

Moreover we set $W_\infty(\cdot) = \inf_{x \in \overline{\Omega}} W(x, \cdot)$ and assume that for any $\epsilon > 0$ there exists a positive constant δ such that

$$(1.3) \quad |W^{1/2}(x, u) - W^{1/2}(y, u)| \leq \epsilon W_\infty^{1/2}(u)$$

for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \delta$ and all $u \in \mathbb{R}^n$. Here from the definition of $W_\infty(u)$ and (1.3) we have the following relation

$$(1.3') \quad |W^{1/2}(x, u) - W^{1/2}(y, u)| \leq \epsilon W^{1/2}(x, u)$$

for all $x, y \in \overline{\Omega}$ with $|x - y| \leq \delta$ and for all $u \in \mathbb{R}^n$.

We think that the conditions (1.1) and (1.3) are not restrictive. In fact, consider continuous functions $W(u)$, $h(x)$, where $W(u)$ satisfies the condition (1.1) and where $h(x)$ is positive function in $\overline{\Omega}$. If the function $W(x, u)$ has a form of $h(x)W(u)$, then we can see that $W(x, u)$ satisfies the conditions (1.1) and (1.3).

In order to state the main theorem, we will introduce a Riemannian metric on \mathbb{R}^n , $d(x, a, b)$ which depends on $x \in \overline{\Omega}$. For $x \in \overline{\Omega}$ and $a, b \in \mathbb{R}^n$, let $d(x, a, b)$ be the metric defined by

$$(1.4) \quad d(x, a, b) = \inf \left\{ \int_0^1 W^{1/2}(x, \gamma(t)) |\dot{\gamma}(t)| dt \mid \gamma \in C^1([0, 1] : \mathbb{R}^n), \right. \\ \left. \gamma(0) = a, \gamma(1) = b \right\}.$$

For example, in the case of $W(x, u) = (u^2 - 1)^2$ and $n = 1$, we have

$$d(x, -1, b) = \int_{-1}^b |s^2 - 1| ds \quad \text{for } b \geq -1.$$

We now state our main theorem of this note.

Theorem 1. (See [6].) Suppose that function W satisfies (1.1) and (1.3) and that g satisfies (1.2). For $\epsilon > 0$, let u_ϵ be a solution of the variational problem:

$$\inf \left\{ \int_{\Omega} [\epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u)] dx \mid u \in W^{1,2}(\Omega : \mathbb{R}^n), u|_{\partial\Omega}(x) = g(x) \right\}.$$

If there exist a positive sequence $\{\epsilon_i\}_{i=1}^{\infty}$ and a function $u_0(x) \in L^1(\Omega : \mathbb{R}^n)$ such that

$$(1.5) \quad \lim_{i \rightarrow \infty} \epsilon_i = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} u_{\epsilon_i} = u_0 \quad \text{in } L^1(\Omega : \mathbb{R}^n),$$

then the function u_0 is characterized by

$$W(x, u_0(x)) = 0 \text{ for almost all } x \in \Omega, \text{ that is, } u_0(x) = \alpha \text{ or } \beta \text{ for almost all } x \in \Omega.$$

Moreover the set $E_0 = \{x \in \Omega \mid u_0(x) = \alpha\}$ is a solution of the variational problem (P_0) :

$$(P_0) \quad \inf \left\{ \int_{\Omega \cap \partial^* E} d(x, \alpha, \beta) d\mathcal{H}_{N-1} + \int_{\partial\Omega} d(x, v|_{\partial\Omega}(x), g(x)) d\mathcal{H}_{N-1} \mid \right. \\ \left. E \subset \Omega, P_{\Omega}(E) < \infty, v = \alpha \chi_E + \beta \chi_{\Omega \setminus E} \right\},$$

where $P_{\Omega}(E)$ is a perimeter of E in Ω and $v|_{\partial\Omega}$ is the trace of v to $\partial\Omega$. Furthermore we have

$$\lim_{i \rightarrow \infty} \int_{\Omega} [\epsilon_i |\nabla u_{\epsilon_i}|^2 + \frac{1}{\epsilon_i} W(x, u_{\epsilon_i})] dx = 2 \int_{\Omega \cap \partial^* E_0} d(x, \alpha, \beta) d\mathcal{H}_{N-1} \\ + 2 \int_{\partial\Omega \cap \partial^* E_0} d(x, \alpha, g(x)) d\mathcal{H}_{N-1} + 2 \int_{\partial\Omega \setminus \partial^* E_0} d(x, \beta, g(x)) d\mathcal{H}_{N-1}.$$

Here $\partial^* E_0$ is the reduced boundary of E_0 .

Remark. It is not restrictive to assume that there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^\infty$ satisfying (1.5). In fact, the following is proved in [4] and [5]: if there exist constants C and R such that

$$(1.6) \quad W_\infty(u) \geq C|u| \quad \text{for } |u| \geq R,$$

then there exists a subsequence $\{u_{\epsilon_i}\}_{i=1}^\infty$ satisfying (1.5).

It is worth noting that the study of asymptotic behavior of minimizer of (P_ϵ) occurs a completely different difficulty from that of (SP_ϵ) . One of the difficulties is that the selection of minimizing sequence $\{\gamma_k\}_{k=1}^\infty$ achieving the value of $d(x, \alpha, g(x))$ depends on the space variable x . In order to overcome this difficulty, we approximate $W(\cdot, u)$ and $g(\cdot)$ by suitable piecewise smooth functions near the transition layer and the boundary $\partial\Omega$.

2. The Main Propositions

At first, we will give functionals F_ϵ and F_0 from $L^1(\Omega : \mathbb{R}^N)$ into $[0, \infty]$. For $u \in L^1(\Omega : \mathbb{R}^n)$ and $\epsilon > 0$, we define $F_\epsilon(u)$, $F_0(u)$ by

$$F_\epsilon(u) = \begin{cases} \int_\Omega [\epsilon |\nabla u|^2 + \frac{1}{\epsilon} W(x, u)] dx, & \text{if } u \in W^{1,2}(\Omega : \mathbb{R}^n) \text{ and } u = g \text{ on } \partial\Omega, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$F_0(u) = \begin{cases} 2 \int_\Omega d(x, \alpha, \beta) |\nabla \chi_{\{u(x)=\alpha\}}| + 2 \int_{\partial\Omega} d(x, u|_{\partial\Omega}(x), g(x)) d\mathcal{H}_{N-1}, \\ \text{if } u \in BV(\Omega : \mathbb{R}^n) \text{ and } W(x, u(x)) = 0 \text{ for almost all } x \in \Omega, \\ +\infty, & \text{otherwise.} \end{cases}$$

In order to prove our main theorem, we need the following two propositions which are crucial in our analysis.

Proposition A. Suppose that $\{v_\epsilon\}_{\epsilon>0}$ is a sequence in $L^1(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \rightarrow 0_+$ to a function v_0 . If

$$\liminf_{\epsilon \rightarrow 0_+} F_\epsilon(v_\epsilon) < +\infty,$$

then v_0 is a function in $BV(\Omega : \mathbb{R}^n)$ such that

$$F_0(v_0) \leq \liminf_{\epsilon \rightarrow 0_+} F_\epsilon(v_\epsilon).$$

Proposition B. Suppose that $w_0 \in L^1(\Omega : \mathbb{R}^n)$ is a function with $w_0 = \alpha\chi_E + \beta\chi_{\Omega \setminus E}$ where E is a measurable subset in Ω with finite perimeter. Then there exists a sequence $\{w_\epsilon\}_{\epsilon > 0}$ in $W^{1,2}(\Omega : \mathbb{R}^n)$ which converges in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \rightarrow 0_+$ to w_0 such that

$$(2.1) \quad \limsup_{\epsilon \rightarrow 0_+} F_\epsilon(w_\epsilon) \leq F_0(w_0).$$

Using Propositions A and B, we can prove Theorem 1 as in the same matter with in [8]. Therefore we have only to prove Proposition A and B. In this note, we will only prove Proposition B for the special case.

On the other hand, in Theorem 1, the minimizers $\{u_\epsilon\}_{\epsilon > 0}$ do not always generate interior layers. For example, if we consider the problem (SP_ϵ) with $g \equiv 0$, we have $E_0 = \Omega$ or \emptyset . In contrast, considering the family of *local* minimizers, from Theorem 1 and the results of [7], we obtain the following theorem.

Theorem 2. Let $u_0 \in L^1(\Omega : \mathbb{R}^n)$ be a isolated L^1 -local minimizer of F_0 , that is,

there exists a positive constant δ such that $F_0(u_0) < F_0(v)$

whenever $u \neq v$ and $\|u_0 - v\|_{L^1(\Omega : \mathbb{R}^n)} \leq \delta$.

Then there exist a constant $\epsilon_0 > 0$ and a sequence $\{u_\epsilon\}_{\epsilon < \epsilon_0}$ such that u_ϵ is a local minimizer of F_ϵ and $u_\epsilon \rightarrow u_0$ in $L^1(\Omega : \mathbb{R}^n)$ as $\epsilon \rightarrow 0$.

3. Proof of Proposition B

In this section, we will only prove Proposition B for the special case that $w_0 \equiv \alpha$ in Ω . In order to prove Proposition B for the case of $w_0 \equiv \alpha$, we need the following two lemmas. The first lemma is obtained easily by the inverse mapping theorem.

Lemma 3–1. Let Ω be a bounded domain with C^2 -smooth boundary $\partial\Omega$. For $x \in \partial\Omega$ let $\nu(x)$ be a inner normal vector to $\partial\Omega$ at x . Define a mapping $\pi : \partial\Omega \times [0, \infty) \rightarrow \mathbb{R}^N$ by

$$(3.1) \quad \pi(x, t) = \pi_t(x) = x + t\nu(x).$$

Then there exists a constant s_0 such that the image of π in $\partial\Omega \times (0, s_0]$ is contained in Ω and the C^1 -smooth inverse mapping π^{-1} of π exists in $\pi(\partial\Omega \times [0, s_0])$.

Lemma 3–2. (See [8] and [9].) Let Ω be an open bounded subset of \mathbb{R}^N with Lipschitz-continuous boundary. Let A be an open subset of \mathbb{R}^N with C^2 , compact, nonempty

boundary such that $\mathcal{H}_{N-1}(\partial A \cap \partial\Omega) = 0$. Define a distance function to ∂A , $d_{\partial A} : \Omega \rightarrow \mathbb{R}$, by $d_{\partial A}(x) = \text{dist}(x, A)$. Then, for some $s_1 > 0$, $d_{\partial A}$ is a C^2 -function in $\{0 < d_{\partial A}(x) < s_1\}$ with

$$(3.2) \quad |\nabla d_{\partial A}| = 1.$$

Furthermore, $\lim_{s \rightarrow 0} \mathcal{H}_{N-1}(\{d_{\partial A}(x) = s\}) = \mathcal{H}_{N-1}(\partial A \cap \Omega)$ and

$$(3.3) \quad |\{x \mid |d_{\partial A}(x)| < s\}| = O(s).$$

By $d_{\partial\Omega}(x)$ we denote a function $\text{dist}(x, \partial\Omega)$. From Lemma 3-2, we can see that $d_{\partial\Omega}$ is a C^2 -function. We set $s^* = \min\{s_0, s_1\}$. For any $\nu \in \mathbb{S}^{N-1}$ we denote by Q_ν the open unit cube centered at the origin with two of its surfaces normal to ν . Furthermore for $x \in \partial\Omega$, $\eta > 0$, and sufficiently small δ with $0 < \delta < s^*$, we set $\partial\Omega_\eta(x) = \partial\Omega \cap (x + \eta Q_{\nu(x)})$ and $\Omega_\eta^\delta(x) = \bigcup_{\delta < t < s^*} \pi_t(\partial\Omega_\eta(x))$.

We will start to prove Proposition B for the special case $w_0 \equiv \alpha$. The proof of Proposition B for the case of $w_0 = \alpha$ requires three steps.

The First Step: Let x_0 be any point in $\partial\Omega$. In this step, for any sufficiently small $\eta > 0$ we will construct a family $\{w_\epsilon^\delta\}_{\epsilon, \delta > 0} \subset W^{1,2}(\Omega_\eta^\delta(x_0) : \mathbb{R}^n)$ such that

$$(3.4) \quad \limsup_{\epsilon, \delta \rightarrow 0} \int_{\Omega_\eta^\delta} [\epsilon |\nabla w_\epsilon^\delta|^2 + \frac{1}{\epsilon} W(x_0, w_\epsilon^\delta)] dx \leq 2d(x_0, \alpha, g(x_0)) \mathcal{H}_{N-1}(\partial\Omega_\eta(x_0)).$$

In this step, for simplicity, we set $\Omega_\eta^\delta = \Omega_\eta^\delta(x_0)$.

In order to construct $\{w_\epsilon^\delta\}_{\epsilon, \delta > 0}$, we fix $\epsilon, \delta > 0$, and consider the following ordinary differential equation:

$$(3.5) \quad \begin{cases} \frac{d}{dt} y_\epsilon(t) = \frac{[\epsilon^{1/2} + W(x_0, \gamma(y_\epsilon(t)))]^{1/2}}{\epsilon |\dot{\gamma}(y_\epsilon(t))|}, \\ y_\epsilon(\delta) = 0. \end{cases}$$

Here by $\dot{\gamma}$ we denote $d\gamma(t)/dt$, and assume that $\gamma \in C^1([0, 1] : [K_1, K_2]^n)$, $\gamma(0) = \alpha$, $\gamma(1) = g(x_0)$. We set

$$\psi_\epsilon(t) = \int_0^t \frac{\epsilon |\dot{\gamma}(t)|}{[\epsilon^{1/2} + W(x_0, \gamma(t))]^{1/2}} dt$$

for $t \in (0, 1)$. Then $\psi_\epsilon(t)$ is a monotone increasing function and

$$(3.6) \quad \tau_\epsilon \equiv \psi_\epsilon(1) \leq \epsilon^{3/4} \cdot \text{length of } \gamma.$$

Here we set $\tilde{y}_\epsilon(t) = \psi_\epsilon^{-1}(t - \delta)$, and we can see that $\tilde{y}_\epsilon(t)$ satisfies (3.5) in $[\delta, \delta + \tau_\epsilon]$ and we define $y_\epsilon(t)$ by

$$(3.7) \quad y_\epsilon(t) \equiv \max\{0, \min\{1, \tilde{y}_\epsilon(t)\}\}.$$

We separate Ω_η^δ to three domains $\Omega_{\eta,i}^\delta$, $i = 1, 2, 3$ as follows:

$$(3.8) \quad \begin{aligned} \Omega_{\eta,1}^\delta &\equiv \{x \in \Omega_\eta^\delta : d_{\partial\Omega}(x) < \delta + \tau_\epsilon, d_S(x) \leq \eta\tau_\epsilon\}; \\ \Omega_{\eta,2}^\delta &\equiv \{x \in \Omega_\eta^\delta : d_{\partial\Omega}(x) < \delta + \tau_\epsilon, d_S(x) \geq \eta\tau_\epsilon\}; \\ \Omega_{\eta,3}^\delta &\equiv \{x \in \Omega_\eta^\delta : d_{\partial\Omega}(x) \geq \delta + \tau_\epsilon\}, \end{aligned}$$

where $d_S(x)$ is a distance function to $\bigcup_{\delta < t < s^*} \pi_t[\partial\Omega \cap (x_0 + \eta\partial Q_{\nu(x_0)})]$. Here we define $w_\epsilon(x)$ on $\bigcup_{i=2,3} \Omega_{\eta,i}^\delta$ as follows:

$$(3.9) \quad w_\epsilon(x) = \begin{cases} \gamma(y_\epsilon(d_{\partial\Omega}(x))), & \text{if } x \in \Omega_{\eta,2}^\delta, \\ \alpha, & \text{if } x \in \Omega_{\eta,3}^\delta. \end{cases}$$

and extend w_ϵ to $\Omega_{\eta,1}^\delta$ such that for any $x \in \Omega_\eta^\delta$ with $d_S(x) = 0$ or $d_{\partial\Omega}(x) = \delta + \tau_\epsilon$, $w_\epsilon(x) = \alpha$ and

$$|\nabla w_\epsilon| \leq 2/(K_2 - K_1)\eta\tau_\epsilon + C/\epsilon \leq C(\eta\tau_\epsilon)^{-1} + C\epsilon^{-1}.$$

For sufficiently small $\epsilon > 0$, we have the length of $\gamma < \epsilon^{-1/8}$ and $\tau_\epsilon \leq \epsilon^{5/8}$. Therefore we obtain

$$(3.10) \quad \begin{aligned} \int_{\Omega_{\eta,1}^\delta} [\epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon}W(x_0, w_\epsilon)]dx &\leq C[\epsilon/\eta^2\tau_\epsilon^2 + 1/\epsilon]\tau_\epsilon^N \mathcal{H}_{N-1}(\partial\Omega_\eta) \\ &\leq C(\epsilon/\eta^2 + \epsilon^{1/4})\tau_\epsilon^{N-2} \mathcal{H}_{N-1}(\partial\Omega_\eta). \end{aligned}$$

Here we note that constants C are independent of ϵ and η . On the other hand, for sufficiently small $\delta > 0$ and $\epsilon > 0$ we have $\delta + \tau_\epsilon < s^* \equiv \min\{s_0, s_1\}$ and obtain from Lemma 3-2 and (3.9)

$$\int_{\bigcup_{i=2,3} \Omega_{\eta,i}^\delta} [\epsilon|\nabla w_\epsilon|^2 + \frac{1}{\epsilon}W(x_0, w_\epsilon)]dx \leq \int_{\Omega_{\eta,2}^\delta} \frac{2}{\epsilon}[\epsilon^{1/2} + W(x_0, \gamma(y_\epsilon(d_{\partial\Omega}(x))))]|\nabla d_{\partial\Omega}(x)|dx,$$

and from the co-area formula in BV functions, we get

$$\begin{aligned}
&\leq 2 \int_{\delta}^{\tau_{\epsilon}+\delta} dt \int_{\Omega_{\eta}^{\delta} \cap \{d_{\partial\Omega}(x)=t\}} \epsilon^{-1} [\epsilon^{1/2} + W(x_0, \gamma(y_{\epsilon}(t)))] d\mathcal{H}_{N-1} \\
&\leq 2\kappa_{\epsilon}^{\delta} \int_{\delta}^{\tau_{\epsilon}+\delta} \epsilon^{-1} (\epsilon^{1/2} + W(x_0, \gamma(y_{\epsilon}(t)))) dt,
\end{aligned}$$

where $\kappa_{\epsilon}^{\delta} = \sup_{\delta \leq d_S(x) \leq \delta+\epsilon} (\Omega_{\eta}^{\delta} \cap \pi_t(\partial\Omega))$. Then from (3.5) we obtain

$$(3.11) \quad \int_{\bigcup_{i=1,2} \Omega_{\eta,i}^{\delta}} [\epsilon |\nabla w_{\epsilon}|^2 + \frac{1}{\epsilon} W(x_0, w_{\epsilon})] dx \leq 2\kappa_{\epsilon}^{\delta} \int_0^1 [\epsilon^{1/2} + W(x_0, \gamma(t))]^{1/2} |\dot{\gamma}(t)| dt.$$

From the regularity of $\partial\Omega$ and the definition of $\Omega_{\eta}^0(x_0)$, there exist a constant η_0 independent of x_0 (dependent only on $\partial\Omega$) such that for any $0 < \eta < \eta_0$, we have $\mathcal{H}_{N-1}(\partial\Omega_{\eta}^0(x_0) \cap \partial\Omega) = 0$. So from Lemma 3-2 we have $\lim_{\epsilon, \delta \rightarrow 0} \kappa_{\epsilon}^{\delta} = \mathcal{H}_{N-1}(\partial\Omega_{\eta}(x_0))$ for any $\eta \in (0, \eta_0)$. Here we set $w_{\epsilon}^{\delta, \gamma} = w_{\epsilon}$. Therefore from (3.10) and (3.11), for any $\eta \in (0, \eta_0)$ we obtain

$$\begin{aligned}
(3.12) \quad &\int_{\Omega_{\eta}^{\delta}(x_0)} [\epsilon |\nabla w_{\epsilon}^{\delta, \gamma}|^2 + \frac{1}{\epsilon} W(x_0, w_{\epsilon}^{\delta, \gamma})] dx \\
&\leq 2\mathcal{H}_{N-1}(\partial\Omega_{\eta}) \int_0^1 W^{1/2}(x_0, \gamma(t)) |\dot{\gamma}(t)| dt \\
&\quad + \mathcal{H}_{N-1}(\partial\Omega_{\eta}) [0(\epsilon/\eta^2) + 0(\epsilon^{1/4}) + 0_{\sqrt{\epsilon^2+\delta^2}}(1)].
\end{aligned}$$

Here by $0_{\epsilon}(1)$ we mean $\lim_{\epsilon \rightarrow 0} 0_{\epsilon}(1) = 0$. Since for any $\epsilon > 0$ there exist a sequence of C^1 -curves $\{\gamma_i\}_{i=1}^{\infty}$ such that the length of $\gamma_i \leq \epsilon^{-1/8}$ and

$$\lim_{i \rightarrow \infty} \int_0^1 W^{1/2}(x_0, \gamma_i(t)) |\dot{\gamma}_i(t)| dt = d(x_0, a, b),$$

by the diagonal argument and (3.12), we can construct a sequence $\{w_{\epsilon}^{\delta}\}_{\epsilon, \delta > 0}$ satisfying (3.4). Therefore the aim of the first step is completed. ■

The Second Step: Let Ω_{δ} be a domain $\{x \in \Omega : \delta < d_{\partial\Omega}(x) < s^*\} = \bigcup_{\delta < t < s^*} \pi_t(\partial\Omega)$. At the second step, we construct a sequence $\{w_{\epsilon}^{\delta}\}_{\epsilon, \delta > 0}$ in $W^{1,2}(\Omega_{\delta}, \mathbb{R}^n)$ such that

$$(3.13) \quad \limsup_{\delta, \epsilon \rightarrow \infty} \int_{\Omega_{\delta}} [\epsilon |\nabla w_{\epsilon}^{\delta}|^2 + \frac{1}{\epsilon} W(x, w_{\epsilon}^{\delta})] dx \leq 2 \int_{\partial\Omega} d(x, a, g(x)) d\mathcal{H}_{N-1}.$$

In order to construct a sequence $\{w_{\epsilon}^{\delta}\}_{\epsilon, \delta > 0}$, we will separate $\partial\Omega$ into small pieces. From the regularity of $\partial\Omega$, for sufficiently small $\eta > 0$, there exist p points $\{x_i\}_{i=1}^p \subset \partial\Omega$ and a subset ω_{η} of $\partial\Omega$ such that

$$(3.14) \quad \partial\Omega \setminus \bigcup_{1 \leq i \leq p} \partial\Omega_{\eta}(x_i) \subset \omega_{\eta}, \quad \partial\Omega_{\eta}(x_i) \cap \partial\Omega_{\eta}(x_j) = \emptyset, \quad i \neq j, \quad i, j = 1, 2, \dots, p$$

and $\lim_{\eta \rightarrow 0} \mathcal{H}_{N-1}(\omega_\eta) = 0$. Here we note that p depends on η and $\lim_{\eta \rightarrow 0} p(\eta) = \infty$.

For any $\eta, \delta, \epsilon > 0$, fix η , δ , and ϵ . Then for any $i \in \{1, 2, \dots, p\}$, from (3.10) we can construct functions $w_\epsilon^{i, \delta, \eta} \in W^{1,2}(\Omega_\eta^\delta(x_i))$ such that

$$(3.15) \quad \begin{aligned} & \int_{\Omega_\eta^\delta(x_i)} [\epsilon |\nabla w_\epsilon^i|^2 + \frac{1}{\epsilon} W(x_i, w_\epsilon^i)] dx \\ & \leq 2\mathcal{H}_{N-1}(\partial\Omega_\eta(x_i)) d(x_i, \alpha, g(x_i)) \\ & \quad + \mathcal{H}_{N-1}(\partial\Omega_\eta(x_i)) [0(\epsilon/\eta^2) + 0(\epsilon^{1/4}) + 0_{\sqrt{\epsilon^2 + \delta^2}}(1)]. \end{aligned}$$

Then we define $w_\epsilon^{\delta, \eta} \in W^{1,2}(\Omega_\delta : \mathbb{R}^n)$ as follows:

$$w_\epsilon^{\delta, \eta} = \begin{cases} w_\epsilon^{i, \delta, \eta}, & \text{if } x \in \Omega_\eta^\delta(x_i), \\ \alpha, & \text{otherwise.} \end{cases}$$

By the argument of Step 1, we can see $w_\epsilon^{\delta, \eta} \in W^{1,2}(\Omega_\delta : \mathbb{R}^n)$ easily. Then we have

$$(3.16) \quad \int_{\Omega_\delta} [\epsilon |\nabla w_\epsilon^{\delta, \eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta, \eta})] dx = \sum_{i=1}^p \int_{\Omega_\eta^\delta(x_i)} [\epsilon |\nabla w_\epsilon^{i, \delta, \eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{i, \delta, \eta})] dx$$

On the other hand, we have (for simplicity we omit the index δ, η of $w_\epsilon^{i, \delta, \eta}$)

$$\begin{aligned} & \int_{\Omega_\eta^\delta(x_i)} [\epsilon |\nabla w_\epsilon^i|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^i)] dx \\ & = \int_{\Omega_\eta^\delta(x_i)} [\epsilon |\nabla w_\epsilon^i|^2 + \frac{1}{\epsilon} W(x_i, w_\epsilon^i)] dx + \int_{\Omega_\eta^\delta(x_i)} \frac{1}{\epsilon} [W(x, w_\epsilon^i) - W(x_i, w_\epsilon^i)] dx \\ & \equiv I_1^i + I_2^i. \end{aligned}$$

From (3.15) we obtain

$$(3.17) \quad \sum_{i=1}^{p(\eta)} I_1^i \leq 2 \sum_{i=1}^{p(\eta)} [d(x_i, \alpha, g(x_i)) \mathcal{H}_{N-1}(\partial\Omega_\eta(x_i))] + 0(\epsilon/\eta^2) + 0(\epsilon^{1/4}) + 0_{\sqrt{\epsilon^2 + \delta^2}}(1),$$

and from (1.2) and (3.15)

$$\sum_{i=1}^{p(\eta)} |I_2^i| \leq \sum_{i=1}^{p(\eta)} \int_{\Omega_\eta^\delta(x_i)} 0_{|x-x_i|}(1) \frac{1}{\epsilon} W(x_i, w_\epsilon^i) dx \leq 0_\eta(1) \sum_{i=1}^{p(\eta)} I_1^i.$$

We set $\eta^2 = \epsilon^{3/4}$. Then combining (3.16) and (3.17), we obtain

$$(3.18) \quad \begin{aligned} & \limsup_{\delta, \epsilon \rightarrow 0} \int_{\Omega_\delta} [\epsilon |\nabla w_\epsilon^{\delta, \eta(\epsilon)}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta, \eta(\epsilon)})] dx \\ & \leq \limsup_{\epsilon \rightarrow 0} 2 \sum_{i=1}^{p(\eta)} d(x_i, \alpha, g(x_i)) \mathcal{H}_{N-1}(\partial\Omega_\eta(x_i)). \end{aligned}$$

From the continuity of the function $d(x, \alpha, g(x))$, we obtain

$$\begin{aligned} \sum_{i=1}^{p(\eta)} d(x_j, \alpha, g(x_j)) \mathcal{H}_{N-1}(\partial\Omega_\eta(x_i)) &\leq \int_{\bigcup_{1 \leq j \leq p} \partial\Omega_\eta(x_i)} d(x, \alpha, g(x)) d\mathcal{H}_{N-1} + 0_\eta(1) \\ &\leq \int_{\partial\Omega} d(x, \alpha, g(x)) d\mathcal{H}_{N-1} + 0_\eta(1). \end{aligned}$$

Therefore combining (3.18), we can see that the sequence $\{w_\epsilon^{\delta, \eta(\epsilon)}\}_{\epsilon, \delta > 0}$ satisfies (3.13).

Hence we set $w_\epsilon^\delta = w_\epsilon^{\delta, \eta(\epsilon)}$, and so the purpose of Step 2 is completed. ■

The Third Step: In this step, we will complete the proof of Proposition B for the special case $w_0 \equiv \alpha$. For any $\delta, \epsilon > 0$ we define w_ϵ^δ as follows:

$$w_\epsilon^\delta = \begin{cases} \alpha, & \text{if } x \in \Omega \setminus \Omega_0, \\ w_\epsilon^{*\delta}, & \text{if } x \in \Omega_\delta \end{cases}$$

where $\Omega_0 = \bigcup_{0 < t < s^*} \pi_t(\partial\Omega)$ and where $w_\epsilon^{*\delta}$ is a function constructed in Step 2. In $\Omega^\delta \equiv \Omega_0 \setminus \Omega_\delta$, we construct w_ϵ^δ by combining between $g(x)$ and $w_\epsilon^{*\delta}(\pi_\delta(x))$ i.e. for $x \in \Omega_0 \setminus \Omega_\delta$,

$$(3.19) \quad w_\epsilon^\delta(x) = \frac{d_{\partial\Omega}(x)}{\delta} w_\epsilon^{*\delta}|_{(\partial\Omega)_\delta}(\pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x)) + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta}\right) g(\pi_{d_{\partial\Omega}(x)}^{-1}(x)).$$

Here $\pi_\delta(x)$ and $\pi_{d_{\partial\Omega}(x)}(x)$ are functions appearing in Lemma 3–1. Then we can see easily $w_\epsilon^\delta \in W^{1,2}(\Omega)$ and $w_\epsilon^\delta(x) = g(x)$ for all $x \in \partial\Omega$.

In order to estimate the gradient of w_ϵ^δ , we fix ϵ, δ , and fix $\{\Omega_\eta^\delta(x_i)\}_{i=1}^p$ and ω_η . Then we set

$$\begin{aligned} (3.20) \quad \Omega_1^\delta &= \{x \in \Omega^\delta : \pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x) \in \bigcup_{1 \leq i \leq p} \partial(\Omega_{\eta,1}^\delta(x_i))\}, \\ \Omega_2^\delta &= \{x \in \Omega^\delta : \pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x) \in \bigcup_{1 \leq i \leq p} \partial(\Omega_{\eta,2}^\delta(x_i))\}, \\ \omega_\eta^\delta &= \bigcup_{0 < t < \delta} \pi_t(\omega_\eta), \end{aligned}$$

and have $\Omega^\delta = \Omega_1^\delta \cup \Omega_2^\delta \cup \omega_\eta^\delta$. Here $\Omega_{\eta,i}^\delta(x)$, $i = 1, 2$ is a domain appearing in Step 1. Furthermore for simplicity, we set

$$\hat{g}(x) = g(\pi_{d_{\partial\Omega}(x)}^{-1}(x)) \quad \text{and} \quad \hat{w}_\epsilon^\delta(x) = w_\epsilon^{*\delta}|_{(\partial\Omega)_\delta}(\pi_\delta \circ \pi_{d_{\partial\Omega}(x)}^{-1}(x))$$

for $x \in \Omega^\delta$. Then from Lemma 3–1 we can see that there exists a constant C such that $|\nabla \hat{g}(x)| \leq C$ for almost all $x \in \Omega^\delta$.

Now in the domains $\Omega_{\eta,1}^\delta$, $\Omega_{\eta,2}^\delta$, and Ω^δ , we will estimate the gradient of w_ϵ^δ , and obtain the inequality (2.1). If $x \in \omega_\eta^\delta$, then from the construction of w_ϵ in Step 2 we see $v_\epsilon^{\delta,\eta} \equiv \alpha$ in a neighborhood of x , and so for almost all $x \in \omega_\eta^\delta$ we have

$$|\nabla w_\epsilon^{\delta,\eta}| \leq C(1 + 1/\delta).$$

So we obtain

$$(3.21) \quad \int_{\omega^\delta} [\epsilon |\nabla w_\epsilon^{\delta,\eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta,\eta})] dx \leq C \left(\frac{\epsilon}{\delta^2} + \epsilon + \frac{1}{\epsilon} \right) \delta \mathcal{H}_{N-1}(\omega).$$

For almost all $x \in \Omega_{\eta,1}^\delta(x_i)$, then we have

$$\begin{aligned} |\nabla w_\epsilon^{\delta,\eta}| &\leq \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} \hat{w}_\epsilon^{\delta,\eta}(x) + \frac{d_{\partial\Omega}(x)}{\delta} |\nabla \hat{w}_\epsilon^{\delta,\eta}(x)| \\ &\quad + \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} \hat{g}(x) + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta} \right) |\nabla \hat{g}(x)|. \end{aligned}$$

Here from the argument in Step 1, there exists a constant C_2 such that $|\nabla v_\epsilon^{\delta,\eta}(x)| \leq C/(\epsilon^{5/8}\eta)$ for all $x \in \Omega_{\eta,1}^\delta$. Moreover we have $|\Omega_{\eta,1}^\delta| \leq C\delta(\epsilon^{5/8}\eta^{N-1})(\mathcal{H}_{N-1}(\partial\Omega)/\eta^{N-1}) \leq C\delta\epsilon^{5/8}$. So we obtain

$$\begin{aligned} (3.22) \quad \int_{\Omega_1^\delta} [\epsilon |\nabla w_\epsilon^{\delta,\eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta,\eta})] dx &\leq C \left[\epsilon \left(\frac{1}{\delta} + \frac{1}{\epsilon^{5/8}\eta} + 1 \right)^2 + \frac{1}{\epsilon} \right] \delta \epsilon^{5/8} \\ &\leq C \left(\frac{\epsilon}{\delta} + \frac{\delta}{\eta^2 \epsilon^{1/4}} + \frac{\delta}{\epsilon} \right) \epsilon^{5/8}. \end{aligned}$$

For any $x \in \Omega_{\eta,2}^\delta(x_i)$, from Step 1 we see $w_\epsilon^*(x) \equiv g(x_i)$ in a neighborhood of x . Then from the Lipschitz continuity of $g(x)$ on $\partial\Omega$ and (3.19) we have

$$\begin{aligned} |\nabla w_\epsilon^{\delta,\eta}| &\leq \frac{|\nabla d_{\partial\Omega}(x)|}{\delta} |g(x_i) - \hat{g}(x)| + \left(1 - \frac{d_{\partial\Omega}(x)}{\delta} \right) |\nabla \hat{g}| \\ &\leq \frac{C}{\delta} |g(x_i) - \hat{g}(x)| + C \leq C \frac{\eta}{\delta} + C. \end{aligned}$$

So we obtain

$$(3.23) \quad \int_{\Omega_2^\delta} [\epsilon |\nabla w_\epsilon^{\delta,\eta}|^2 + \frac{1}{\epsilon} W(x, w_\epsilon^{\delta,\eta})] dx \leq C \left(\epsilon \left(\frac{\eta}{\delta} \right)^2 + \epsilon + \frac{1}{\epsilon} \right) \delta \mathcal{H}_{N-1}(\partial\Omega).$$

Let $\sigma(\cdot)$ be a positive function with $\sigma(0) = 0$ such that $\lim_{\epsilon \rightarrow 0} \mathcal{H}_{N-1}(\omega_{\eta(\epsilon)})/\sigma(\epsilon) = 0$ and $\lim_{\epsilon \rightarrow 0} \epsilon^{5/8}/\sigma(\epsilon) = 0$. Here we set $\delta_\epsilon = \epsilon\sigma(\epsilon)$, and define $w_\epsilon = w_\epsilon^{\delta_\epsilon}$. Then from (3.21)–(3.23) we obtain

$$(3.24) \quad \lim_{\epsilon \rightarrow 0} \int_{\Omega_{\epsilon\sigma(\epsilon)}} [\epsilon |\nabla w_\epsilon|^2 + \frac{1}{\epsilon} W(x, w_\epsilon)] dx = 0.$$

Therefore from (3.13) and (3.24) we obtain

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega} [\epsilon |\nabla w_{\epsilon}|^2 + \frac{1}{\epsilon} W(x, w)] dx \leq 2 \int_{\partial\Omega} d(x, \alpha, g(x)) d\mathcal{H}_{N-1}.$$

Hence the proof of Proposition *B* for the special case $w_0 = \alpha$ is completed. ■

Finally we remark that the proof of this section is an essential part of complete proof of Proposition *B*.

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